Interface Homogenization Technique for Electromagnetic Finite Element Analysis Including Anisotropic Media

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In some electromagnetic finite-element applications, using a mesh that is not fitted to the geometry of the analyzed objects can provide substantial advantages over a standard fitted mesh in that, for example, moving objects are efficiently handled with no mesh reconstruction. Unfortunately, an unfitted mesh around the interfaces generally leads to a large numerical error. In this study, we present an enhanced version of the interface homogenization technique, which is a recently proposed technique that improves the accuracy of the unfitted finite element analysis, to enable anisotropic media to be handled.

***Index Terms*—Finite element analysis, Electrostatics, Magnetostatics, Magnetic anisotropy.**

I. Introduction

IN finite element (FE) analysis for electromagnetic problems, to handle accurately various geometries of the analyzed objects, one generally generates a mesh (computational grid) that is precisely fitted to the interfaces of the different media (Fig. 1(a)). However, using an unfitted mesh (Fig. 1(b)) can provide several advantages in that, for example, one can handle moving interfaces with no mesh reconstruction, or that it becomes possible to use a structured mesh, which significantly reduces the cost of mesh generation. Because an unfitted mesh, as might be expected, reduces the accuracy of the analysis, an effort has been made to analyze or improve the accuracy of unfitted FE analysis [1]–[4].

We recently proposed a new technique, called interface homogenization (IH) [5], for improving the accuracy of unfitted FE analysis. The IH technique provides an optimum representation for the fields that are uniform in each material, and requires no modification of the FE algorithm, except for the manner in which the material properties are determined. The test analyses in [5] suggest that the IH technique can achieve an optimal order convergence of the FE solution when the unfitted interfaces are sufficiently smooth.

Whereas [5] presents a concrete procedure of the IH technique with the proviso that all media in the analysis domain are isotropic, it is not straightforward to extend the technique for anisotropic media. In this study, we present a more organized procedure of the IH technique, and thereby enable anisotropic media to be handled.

II. FE Analysis

For ease in explanation, consider the two-dimensional electrostatic problem where the electric scalar potential is given by $\phi = \phi\,(x, y)$ and there is no electric charge in the analysis domain. The technique described below, however, can be applied to two-dimensional/three-dimensional magnetostatic analysis involving magnetic anisotropy, in almost the same manner.

The basic equation of the problem is

$$-\nabla \cdot (\varepsilon \nabla \phi) = 0\,, \tag{1}$$

where ε is the electric permittivity. Anisotropic media may be present in the analysis domain.

The standard FE formulation for (1) leads to a linear system of equations:

$$K\boldsymbol{\varphi} = \boldsymbol{f}\,, \tag{2}$$

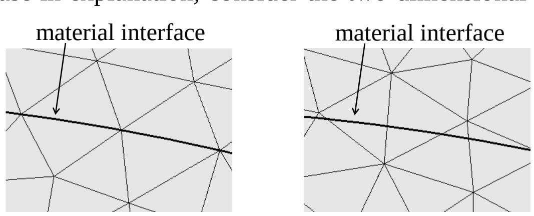
with a coefficient matrix denoted by K, unknowns by $\boldsymbol{\varphi}$, and a right-hand-side vector by $\boldsymbol{f}$. The elements of $\boldsymbol{f}$ are zero, except for the terms relating to the boundary conditions. The entries of K are given by

$$[K]_{ij} = \int \nabla N_i \cdot (\varepsilon \nabla N_j)\, d\Omega\,. \tag{3}$$

Here, N_i denotes the FE shape function, which is assumed to be first order. The FE solution is given by $\phi^{\mathrm{FEM}} = \Sigma \phi_i N_i$, where ϕ_i (i-th entry of $\boldsymbol{\varphi}$) gives the approximation of ϕ at i-th node.

III. Interface Homogenization for Anisotropic Media

According to [5], consider a simple situation of a flat interface formed between two different media and the electric fields are uniform in each of the two media. Because a more general situation locally approaches this simple situation through mesh refinement, the accuracy of the FE analysis for the latter has a significant effect on the convergence of the FE



(a) Fitted mesh (b) Unfitted mesh

Fig. 1. Fitted and unfitted meshes.

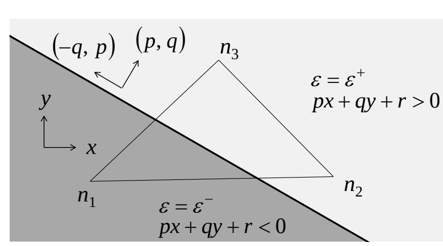
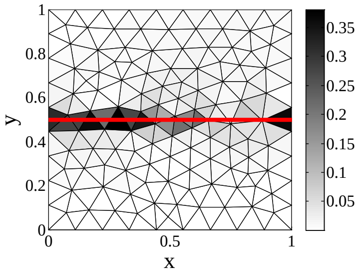
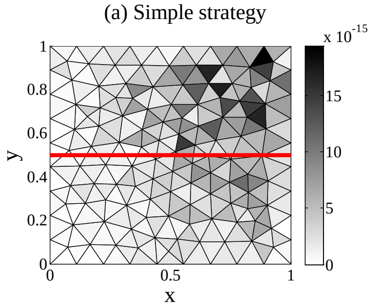


Fig. 2. Unfitted element that lies across a material interface.



(a) Simple strategy



(b) Interface homogenization (proposed technique)

Fig. 3. Error in the electric field provided by unfitted FE analyses.

solution. Unfortunately, despite the simplicity of the situation, the standard FE analysis can produce large errors around the interface, unless the mesh used is fitted to the interface [5]. The IH technique presented below gives a remedy to this problem.

Fig. 2 shows an unfitted element on a material interface. Our goal is to determine the homogenized electric permittivity of the unfitted element, so as to guarantee that, in the simple situation mentioned above, the FE analysis provides the exact solution (in exact arithmetic).

Let the line segment

$$l(x, y) = px + qy + r = 0 \tag{4}$$

be the material interface. The electric field strength and electric flux density are written

$$\boldsymbol{E} = \begin{cases} \boldsymbol{E}^+ & (l > 0) \\ \boldsymbol{E}^- & (l < 0) \end{cases}, \quad \boldsymbol{D} = \begin{cases} \boldsymbol{D}^+ = \varepsilon^+ \boldsymbol{E}^+ & (l > 0) \\ \boldsymbol{D}^- = \varepsilon^- \boldsymbol{E}^- & (l < 0) \end{cases}, \tag{5}$$

with the electric permittivity

$$\varepsilon = \begin{cases} \varepsilon^+ & (l > 0) \\ \varepsilon^- & (l < 0) \end{cases}. \tag{6}$$

The standard FE calculation obtains the electric field within the element using

$$\boldsymbol{E}^{\mathrm{FEM}} = -\frac{1}{2S} \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{7}$$

with

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{pmatrix}, \tag{8}$$

$$S = \frac{1}{2} \sum a_i \, . \tag{9}$$

Here, (x_i, y_i) denote the coordinates of the three nodes.

Assuming ϕ_i is equal to the exact potential that is consistent with (5), if

$$\int_e \nabla N_i \cdot \left(\varepsilon \boldsymbol{E}^{\mathrm{FEM}}\right) d\Omega = \int_e \nabla N_i \cdot \boldsymbol{D} \, d\Omega \tag{10}$$

is satisfied in each element, (2) holds exactly for the simple situation (In other words, the FE and the exact solutions are identical). Note that, in the simple situation, the vector assembly of the right-hand side of (10) invariably results in zero.

One can ensure (10) by determining the homogenized electric permittivity as below (In this short paper, we omit significant details of the calculation.):

$$\varepsilon^{\mathrm{IH}} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}^{-1}, \tag{11}$$

with

$$\alpha = \left(S^+ \varepsilon^+_{tn} / \varepsilon^+_{nn} + S^- \varepsilon^-_{tn} / \varepsilon^-_{nn}\right) / S \, , \tag{12}$$

$$\beta = \left[S^+ \left(\varepsilon^+_{tt} - \varepsilon^+_{nt} \varepsilon^+_{tn} / \varepsilon^+_{nn}\right) + S^- \left(\varepsilon^-_{tt} - \varepsilon^-_{nt} \varepsilon^-_{tn} / \varepsilon^-_{nn}\right)\right] / S \, , \tag{13}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{2S} \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} l_1 / \varepsilon^1_{nn} \\ l_2 / \varepsilon^2_{nn} \\ l_3 / \varepsilon^3_{nn} \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\frac{1}{2S} \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} l_1 \varepsilon^1_{nt} / \varepsilon^1_{nn} \\ l_2 \varepsilon^2_{nt} / \varepsilon^2_{nn} \\ l_3 \varepsilon^2_{nt} / \varepsilon^3_{nn} \end{pmatrix} + \begin{pmatrix} -q \\ p \end{pmatrix}. \tag{15}$$

Here, ε^i is the electric permittivity of the medium in which node i lies, the subscripts n and t signify the tensor components in the local coordinate system for which the bases are $(p, q)^{\mathrm{T}}$ and $(-q, p)^{\mathrm{T}}$, $S^{\pm}$ denotes the area of the sub-region in which $\varepsilon = \varepsilon^{\pm}$, and $l_i = px_i + qy_i + r$.

Fig. 3 shows the errors with respect to the electric field strength that are obtained by the unfitted FE analyses for the simple situation (line $y = 0.5$ is the interface), where

$$\varepsilon^+ = \begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix}, \quad \varepsilon^- = \begin{pmatrix} 15 & 0 \\ 0 & 5 \end{pmatrix}. \tag{16}$$

The IH technique removes the error perfectly that arise around the unfitted interface when using a simple strategy (i.e., determining the element-wise electric permittivity according to the position of the center of gravity for each triangle).

More details of the technique as well as additional numerical results will appear in the full paper.

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